Adapted two step peer methods for advection diffusion problems

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Research topics

- Stochastic problems
- Oscillatory problems
  - Quadrature problems with oscillating integrand
  - Ordinary differential equations with oscillatory solution
- Evolutionary problems with memory
  - Volterra integral and integro-differential equations
  - Fractional equations
- Nonlinear stability issues for ordinary differential equations
  - Algebraically stable general linear methods
  - Strong stability preserving general linear methods

Collaborations

Beatrice Paternoster and *Angelamaria Cardone*, Univ. Salerno; *Raffaele D'Ambrosio*, Univ. L'Aquila; *Giovanni Capobianco*, Univ. Molise; *Liviu Gr. Ixaru*, Institute of Physics and Nuclear Engineering, Bucharest; *Zdzislaw Jackiewicz*, Arizona State Univ.; *Leila Moradi*, *Fakhrodin Mohamadi*, Univ. Hormozgan, Iran; *Sedaghat Shahmorad*, *Ali Abdi*, Univ. Tabriz, Iran.

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   - Improved stochastic ϑ-method
   - Stability regions and numerical tests
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   - Problem and some applications
   - Adapted numerical modeling
   - Classical peer methods
   - Adapted peer methods
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   - Adapted IMEX peer methods for advection-diffusion problems
3. **Conclusions and future work**

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Problem: Stochastic Volterra integral equations

$$X_t = X_0 + \int_0^t a(t, s, X_s)ds + \int_0^t b(t, s, X_s)dW_s, \quad t \in [0, T],$$

- a and b measurable functions
- X_0 random variable
- Itô integral with respect to the Brownian motion W_s
- X_t is random variable for each t.

Some applications:

- economy (general stock, insurance, portfolio, financial markets) [S. Vahdati, 2017; Zhao, Q. *et al.*, 2016; Agram, N. *et al.*, 2015];
- engineering (vibrations of a beam) [Zakes, F. *et al.*, 2018]

A model from economics*

$$X_t = G_t + \int_0^t a(t,s)X_s ds + \int_0^t b(t,s)X_s dW_s, \quad t \in [0,T],$$

$$G_t = \int_0^\infty \phi_s e^{-\int_0^t m(s+r)dr} p_0(t+s)ds + \epsilon \int_t^\infty \phi_{v-t} e^{-\int_0^t m(r+v-t)dr} dW_v,$$

$$a(t,s) = p_0(t-s)e^{-\int_0^{t-s} m(r)dr}, \quad b(t,s) = \epsilon e^{-\int_0^{t-s} m(r)dr}, \quad 0 \le s \le t.$$

- X_t capital density at time t;
- ϕ_s initial capital density resulting from investments that have age s;
- $m(r) \ge 0$ age-dependent relative death rate of the equipment or of the machines involved in the production
- $P_t = p_0(t) + \epsilon W_t$ with $p_0(t) = \mathbb{E}[P_t]$, $\epsilon > 0$ productivity function subject to random fluctuations.

*[Zhao et al. 2016; Agram and Oksendal 2015; Holden, Oksendal et al. 2010]

Stochastic Volterra integral equations

$$X_t = X_0 + \int_0^t a(t, s, X_s)ds + \int_0^t b(t, s, X_s)dW_s, \quad t \in [0, T],$$

- a and b measurable functions
- X_0 random variable
- Itô integral with respect to the Brownian motion W_s
- X_t is random variable for each t.

Some literature:

- **collocation-methods** [Cao et al. 2015, Mirzaee et al. 2014, Xiao et al. 2018];
- **wavelet-based methods** [Heydari et al. 2018, Mohammadi et al. 2016];
- **Euler-Maruyama/Milstein methods** [Wen et al. 2009 and 2011];
- **stochastic ϑ-methods** [Conte, D'Ambrosio, Paternoster 2018; Cardone, Conte, D'Ambrosio, Paternoster 2018].

$$X_t = X_0 + \int_0^t a(t, s, X_s)ds + \int_0^t b(t, s, X_s)dW_s, \quad t \in [0, T],$$

Stochastic ϑ-method [Conte, D'Ambrosio, Paternoster, 2018]

Let $Y_0 = X_0$ and $h = t_{i+1} - t_i$.

$$Y_n = Y_0 + h\sum_{i=0}^{n-1}(\vartheta a(t_n, t_{i+1}, Y_{i+1}) + (1-\vartheta)a(t_n, t_i, Y_i)) + \sqrt{h}\sum_{i=0}^{n-1} b(t_n, t_i, Y_i)V_i,$$

where V_i is a standard Gaussian random variable: $\mathcal{N}(0,1)$-distributed.

Theorem: order of convergence

Under suitable regularity assumptions on the coefficients a and b there exist a constant C such that

$$\mathbb{E}[(X_n - Y_n)^2] \leq Ch.$$

Stability analysis

- Basic test equation:

$$X_t = X_0 + \int_0^t \lambda X_s \, ds + \int_0^t \mu X_s \, dW_s, \quad \lambda, \mu \in \mathbb{R};$$

- convolution test equation:

$$X_t = X_0 + \int_0^t (\lambda + \sigma(t-s)) \, X_s \, ds + \int_0^t \mu X_s \, dW_s, \quad \lambda, \mu, \sigma \in \mathbb{R}.$$

Mean-square stability of exact solutions:

$$\lim_{t \to \infty} \mathbb{E}\left[X_t^2\right] = 0.$$

Mean-square stability of numerical method:

$$\lim_{n \to \infty} \mathbb{E}\left[Y_n^2\right] = 0.$$

Stability of the exact solutions

Basic test equation [D.J. Higham, 2001]

$$X_t = X_0 + \int_0^t \lambda X_s \, ds + \int_0^t \mu X_s \, dW_s, \quad \lambda, \mu \in \mathbb{R};$$

Mean square stability of solutions if and only if

$$\mathrm{Re}(\lambda) + \frac{1}{2}|\mu|^2 < 0.$$

Convolution test equation [Conte, D'Ambrosio, Paternoster, submitted]

$$X_t = X_0 + \int_0^t (\lambda + \sigma(t-s)) \, X_s \, ds + \int_0^t \mu X_s \, dW_s, \quad \lambda, \mu, \sigma \in \mathbb{R}.$$

Mean square stability of solutions if and only if

$$\mathrm{Re}(\lambda) < 0 \quad \mathrm{Re}(s_i) < 0$$

being s_i the roots of

$$q(s) = s^3 - (3\lambda + \mu^2)s^2 + (2\lambda^2 + \lambda\mu^2 - 4\sigma)s + 2\sigma\mu^2 + 4\sigma\lambda.$$

Stability analysis of ϑ-method with respect to basic test equation

Theorem (Recurrence relation)

Let $x = h\lambda$ and $y = h\mu^2$. Then

$$Y_{n+1} = (\alpha + \beta V_n) Y_n,$$

with

$$\alpha = \frac{1 + x(1-\vartheta)}{1 - \vartheta x}, \quad \beta = \frac{\sqrt{y}}{1 - \vartheta x}.$$

Theorem (Mean-square stability)

The stochastic ϑ-method is mean-square stable if and only if

$$\left|\alpha^2 + \beta^2\right| < 1.$$

Stability analysis of ϑ-method with respect to convolution test equation

Theorem (Recurrence relation)

Let $x = h\lambda$, $y = h\mu^2$, $z = h^2\sigma$, $\mu = 2 + (1-2\vartheta)x + z$, $\nu = 1 + (1-\vartheta)x$. *Then*

$$(1-\vartheta x)Y_{n+2} = (\mu + \sqrt{y}V_{n+1})Y_{n+1} - (\nu + \sqrt{y}V_n)Y_n,$$

with

$$(1-\vartheta x)Y_1 = (1 + (1-\vartheta)x + (1-\vartheta)z + \sqrt{y}V_0)Y_0.$$

Theorem (Mean-square stability)

The stochastic ϑ*-methods is mean-square stable if* $\rho(K) < 1$, *with*

$$K = \begin{bmatrix} 0 & 0 & 1 \\ -\dfrac{y}{(1-\vartheta x)^2} & -\dfrac{\nu}{1-\vartheta x} & \dfrac{\mu}{1-\vartheta x} \\ \dfrac{(\nu^2+y)(1-\vartheta x) - 2\mu y}{(1-\vartheta x)^3} & -\dfrac{2\nu\mu}{(1-\vartheta x)^2} & \dfrac{\mu^2+y}{(1-\vartheta x)^2} \end{bmatrix}.$$

Stochastic ϑ-method

Let $Y_0 = X_0$ and $h = t_{i+1} - t_i$.

$$Y_n = Y_0 + h\sum_{i=0}^{n-1}(\vartheta a(t_n, t_{i+1}, Y_{i+1}) + (1-\vartheta)a(t_n, t_i, Y_i)) + \sqrt{h}\sum_{i=0}^{n-1} b(t_n, t_i, Y_i)V_i,$$

where V_i is a standard Gaussian random variable: $\mathcal{N}(0,1)$-distributed.

- same order of convergence of Euler-Maruyama method for SVIEs
- better stability properties with respect to both basic and convolution test equations

Connection with Stochastic Differential Equations (SDEs)

Basic test equation

- basic test equation for SVIEs:

$$X_t = X_0 + \int_0^t \lambda X_s \, ds + \int_0^t \mu X_s \, dW_s \tag{1}$$

equivalent to linear test equation for SDEs:

$$dX_t = \lambda X_t dt + \mu X_t dW_t \tag{2}$$

- ϑ-method for SVIEs applied to (1) corresponds to ϑ-method for SDEs applied to (2).

Connection with Stochastic Differential Equations (SDEs)

Convolution test equation

- convolution test equation for SVIEs:

$$X_t = X_0 + \int_0^t (\lambda + \sigma(t-s))\, X_s\, ds + \int_0^t \mu X_s\, dW_s \tag{3}$$

equivalent to linear 2 by 2 linear system of SDEs:

$$d\begin{bmatrix} X_t \\ Z_t \end{bmatrix} = A \begin{bmatrix} X_t \\ Z_t \end{bmatrix} dt + B \begin{bmatrix} X_t \\ Z_t \end{bmatrix} dW_t, \tag{4}$$

with

$$A = \begin{bmatrix} \lambda & \sigma \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} \mu & 0 \\ 0 & 0 \end{bmatrix}, \qquad Z_t = \int_0^t X_s ds$$

.

- ϑ-method for SVIEs applied to (3) different from ϑ-method for SDEs applied to (4)
- ϑ-method for SDEs provides larger stability regions

AIM

Improve stability properties of stochastic ϑ-method by inheriting better stability properties from SDEs

HOW

Novel quadrature formula for the deterministic integral in the SVIE

Improved stochastic ϑ-method

Let $Y_0 = X_0$ and $h = t_{i+1} - t_i$.

$$Y_n = Y_0 + h\sum_{i=0}^{n-1}(\vartheta a(t_n, t_{i+1} - \vartheta h, Y_{i+1}) + (1-\vartheta)a(t_n, t_i + (1-\vartheta)h, Y_i))$$
$$+ \sqrt{h}\sum_{i=0}^{n-1} b(t_n, t_i, Y_i)V_i.$$

where V_i is a standard Gaussian random variable: $\mathcal{N}(0,1)$-distributed.

- same order of convergence of stochastic ϑ-method for SVIEs
- same stability properties with respect to basic test equation
- better stability properties with respect to convolution test equation: same recurrence relation of ϑ-method for SDEs applied to the 2 by 2 linear system of SDEs (4)

Stability analysis of improved ϑ-method with respect to convolution test equation

Theorem (Recurrence relation)

Let $x = h\lambda$, $y = h\mu^2$, $z = h^2\sigma$, $\tilde{\mu} = 2 + (1-2\vartheta)x + 2\vartheta(1-\vartheta)z$, $\tilde{\nu} = 1 + (1-\vartheta)x - (1-\vartheta)^2 z$. *Then*

$$(1 - \vartheta x - \vartheta^2 z) Y_{n+2} = (\tilde{\mu} + \sqrt{y} V_{n+1}) Y_{n+1} - (\tilde{\nu} + \sqrt{y} V_n) Y_n,$$

with

$$(1 - \vartheta x - \vartheta^2 z) Y_1 = (1 + (1-\vartheta)x + \vartheta(1-\vartheta)z + \sqrt{y} V_0) Y_0.$$

Theorem (Mean-square stability)

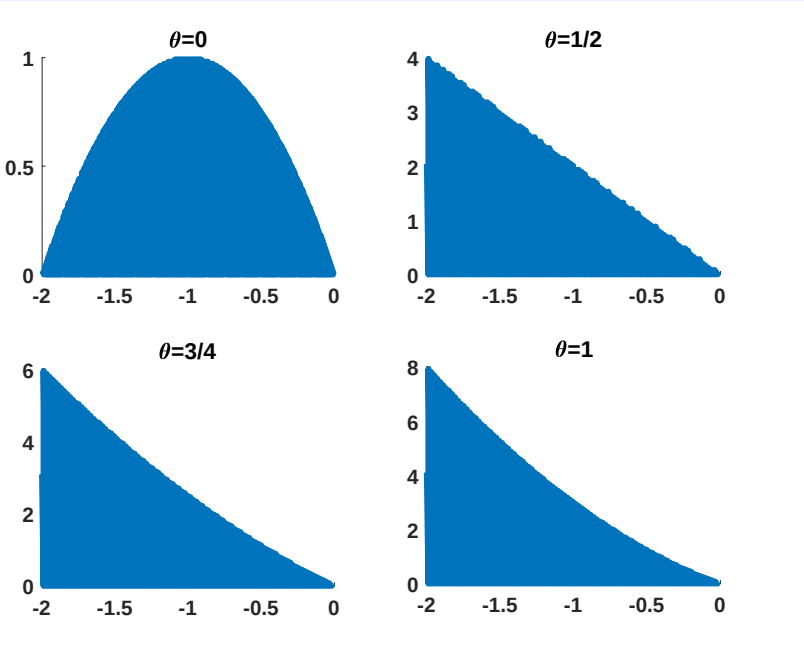
The improved stochastic ϑ-methods is mean-square stable if $\rho(\tilde{K}) < 1$, with

$$\tilde{K} = \begin{bmatrix} 0 & 0 & 1 \\ -\dfrac{y}{(1-\vartheta x - \vartheta^2 z)^2} & -\dfrac{\tilde{\nu}}{1-\vartheta x - \vartheta^2 z} & \dfrac{\tilde{\mu}}{1-\vartheta x - \vartheta^2 z} \\ \dfrac{(\tilde{\nu}^2 + y)(1-\vartheta x - \vartheta^2 z) - 2\tilde{\mu} y}{(1-\vartheta x - \vartheta^2 z)^3} & -\dfrac{2\tilde{\nu}\tilde{\mu}}{(1-\vartheta x - \vartheta^2 z)^2} & \dfrac{\tilde{\mu}^2 + y}{(1-\vartheta x - \vartheta^2 z)^2} \end{bmatrix}.$$

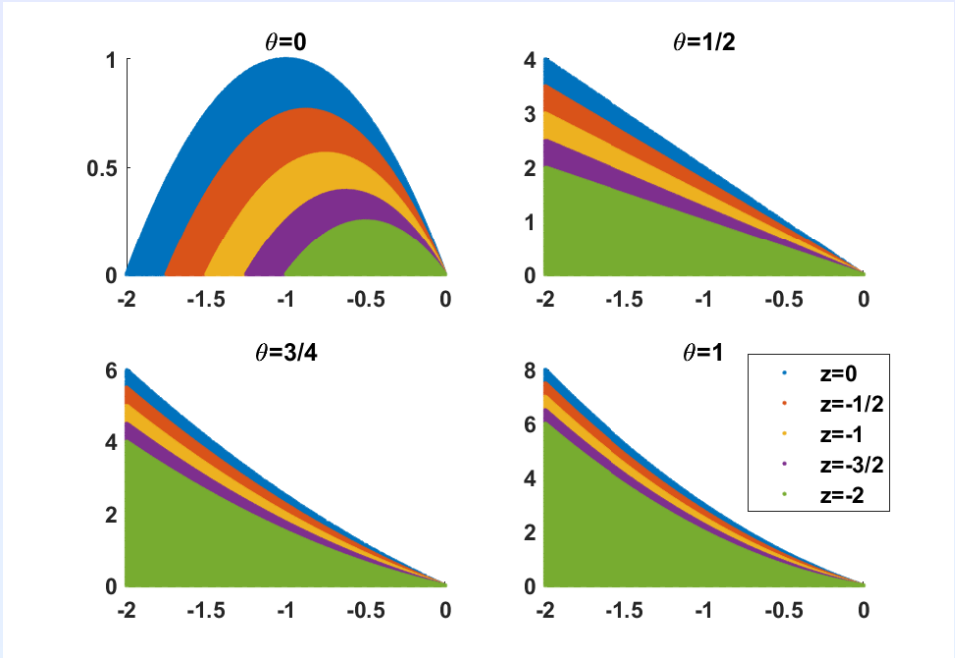
Stability regions for stochastic ϑ-method

- $x = h\lambda$, $y = h\mu^2$, $z = h^2\sigma$
- Stability regions in the plane (x, y)
- Euler-Maruyama method corresponds to $\vartheta = 0$

Basic test equation



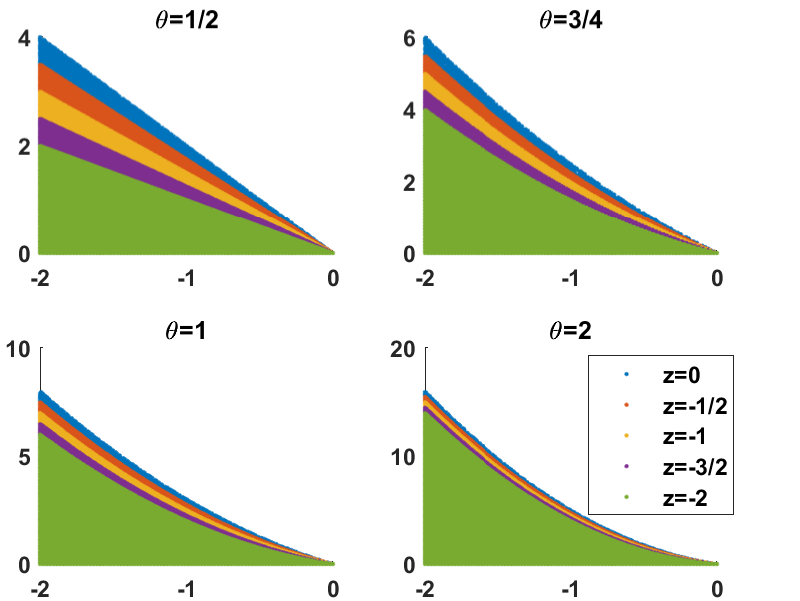
Convolution test equation



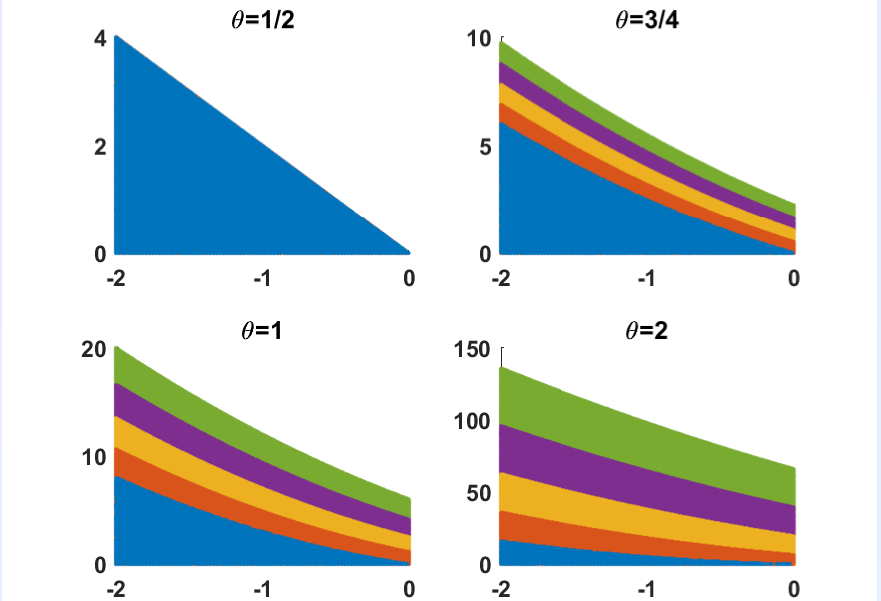
Stability regions with respect to convolution test equation

- $x = h\lambda$, $y = h\mu^2$, $z = h^2\sigma$
- Stability regions in the plane (x, y)

ϑ-method



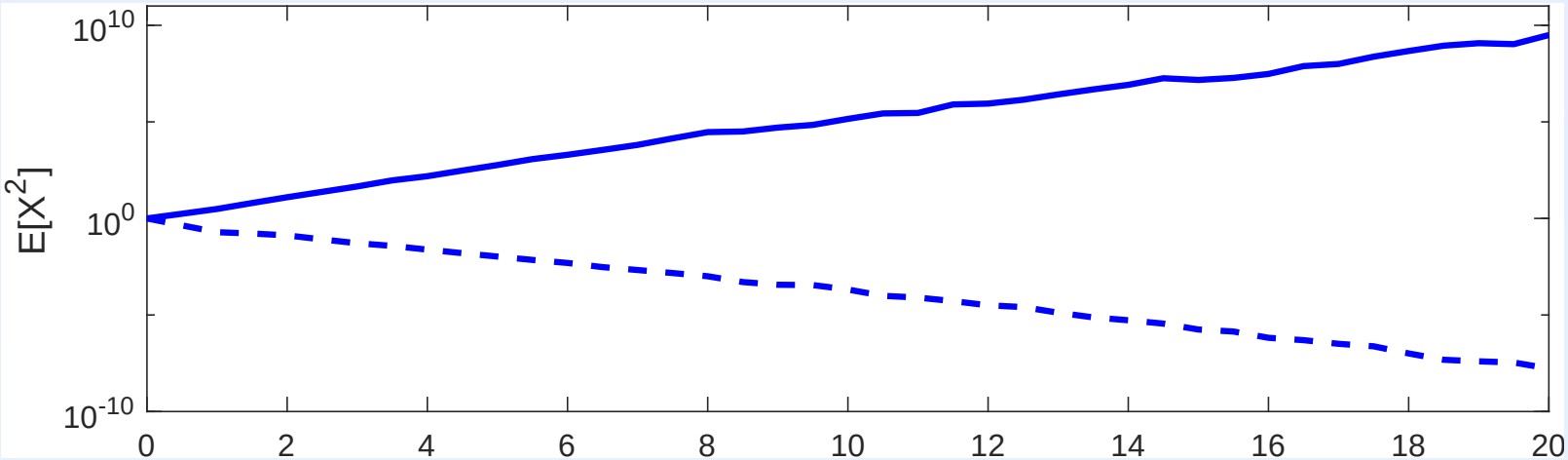
improved ϑ-method



Numerical evidence

$$X_t = X_0 + \int_0^t (\lambda + \sigma(t-s))\, X_s\, ds + \int_0^t \mu X_s\, dW_s, \quad \lambda, \mu, \sigma \in \mathbb{R}.$$

- $\lambda = -2$, $\mu = 2\sqrt{3}$, $\sigma = -8$;
- mean-square of the numerical solution over 1000 realizations with stepsize $h = 1/2$, $\vartheta = 1$;
- $(x, y, z) = (h\lambda, h\mu^2, h^2\sigma) = (-1, 6, -2)$:
  - outside the stability region of ϑ-method,
  - inside the stability region of improved ϑ-method;
- ϑ-method: solid line, improved ϑ-method: dashed line.



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Problem

ODEs with oscillatory solution arising in the semi-discretization in space of PDEs:

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0, \quad t \in [t_0, T], \quad f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d.$$

Some applications

Reaction-diffusion problems:

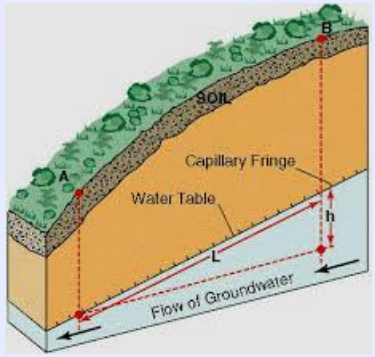
- calcium signalling [Atri *et al.*, 1993; Sherrat *et al.*, 1996];
- cell cycles [Ferrell *et al.*, 2011; Murray *et al.*, 1989];
- hydrodynamics [Su *et al.*, 2003].

Advection-diffusion problems:

- groundwater flows on a sloping impervious base, [Marino *et al.*, 1982; Logan *et al.*, 1995; Su *et al.*, 2003].

A model from groundwater flows on a sloping impervious base

- Groundwater flows on a sloping impervious base is important in dynamic interactions between aquifers and the sea in coastal regions.



- Model by Marino *et al.* (1982), Werner *et al.* (1958), Su *et al.* (2003)
- the **Boussinesq equation** describes the height h of the water table

$$\frac{\partial h}{\partial t} = \frac{K}{S}(h\frac{\partial^2 h}{\partial x^2} + (\frac{\partial h}{\partial x})^2 - \vartheta\frac{\partial h}{\partial x}),$$

where S is the drainable porosity, K is the hydraulic conductivity and ϑ is the slope of the impervious base.

Advection-diffusion problem

If h shows a small deviation from the weighted depth, by setting $\gamma = \frac{T}{S}$, $\nu = K\frac{\vartheta}{S}$, where T is the transmissivity, the model assumes the form:

Boussinesq equation [Su *et al.* 2003]

$$
\begin{aligned}
&h_t(x,t) = \gamma h_{xx}(x,t) - \nu h_x(x,t), \quad (x,t) \in (0,X) \times (0,T)\\
&h(x,0) = h_0(x), \quad x \in [0,X],\\
&h(0,t) = h(X,t) = f(t), \quad t \in [0,T],
\end{aligned}
$$

- $h_0(x)$: initial watertable
- $f(t)$: periodic vertical perturbation relative to the mean sea level due to tidal waves

Advection-diffusion problem

Profile of the solution [Su *et al.*, 2003]

If Boussinesq equation is subject to the periodic boundary condition

$$f(t) = \exp(i\omega t),$$

the solution exhibits the following form:

$$h(x,t) = \exp[(\frac{\nu}{2D} - \mu)x] \exp[i(\omega t - \rho x)]$$

where $\mu = \frac{1}{2\gamma}\sqrt{2\gamma\sqrt{\omega^2 + \frac{\nu^4}{16\gamma^2}} + \frac{\nu^2}{2\gamma}}$, $\rho = \frac{1}{2\gamma}\sqrt{2\gamma\sqrt{\omega^2 + \frac{\nu^4}{16\gamma^2}} - \frac{\nu^2}{2\gamma}}$.

- oscillation both in space and in time

Adapted numerical modeling

Classical numerical methods

- exact on polynomial basis $\{1, t, t^2, ...\}$
- constant coefficients

Adapted numerical methods

Exponential Fitting

$$\mathcal{F} = \{1, t, t^2, \ldots, e^{\pm\mu t}, te^{\pm\mu t}, t^2 e^{\pm\mu t}, \ldots\}$$

$\mu = \mathrm{i}\omega$ for oscillatory problems

- function fitting: exact on non-polynomial basis
- non-constant coefficients, depending on parameters (e.g. μ, ω)

- chose suitable fitting space $\mathcal{F}$
- compute/estimate the parameters

Some literature on Exponential Fitting

- interpolation, numerical differentiation and quadrature

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- ordinary differential equations

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Ordinary differential equations with oscillatory solution

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0, \quad t \in [t_0, T], \quad f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d.$$

Classical peer methods [Schmitt et al. 2004, Soleimani et al. 2017, Weiner et al. 2008]

$$Y_{n+1} = (B \otimes I_d)\, Y_n + h\,(A \otimes I_d)\, F(Y_n) + h\,(R \otimes I_d)\, F(Y_{n+1})$$

$$Y_n = [Y_{ni}]_{i=1}^s\,, \; F(Y_n) = [f(t_{ni}, Y_{ni})]_{i=1}^s$$

$$Y_{ni} \approx y(t_{ni}), \quad t_{ni} = t_n + c_i\, h, \quad i = 1, \ldots, s \quad n = 0, \ldots, N-1 \quad c_s = 1$$

Advantages

- peer variables sharing the same accuracy properties: existence of a continuous extension by interpolation, no order reduction for stiff systems;
- R diagonal matrix: inherent 'method parallelism', useful for large systems arising in semidiscretization of PDEs;
- R lower triangular: explicit methods (can choose $R = 0$).

$$Y_{n+1} = (B \otimes I_d)\, Y_n + h\,(A \otimes I_d)\, F(Y_n) + h\,(R \otimes I_d)\, F(Y_{n+1})$$

Theorem: Order conditions for classical peer methods [Schmitt et al. 2004, Soleimani et al. 2017, Weiner et al. 2008]

The peer method has order $p = s$ if

$$B\,\mathbf{1} = \mathbf{1}, \tag{5a}$$

$$A \;\; = (E_1 - B\,E_2 - R\,E_4)\; E_3^{-1}, \tag{5b}$$

with $\mathbf{1} = [1, 1, \ldots, 1]^T$,

$$E_1 = C\,V_0, \quad E_2 = (C - I_s)\,V_1, \quad E_3 = V_1\,D, \quad E_4 = V_0\,D,$$

where $C = diag(c_i)_{i=1}^{s}$, $D = diag(1, ..., s)$, $V_0 = \left(c_i^{j-1}\right)_{i,j=1}^{s}$,
$V_1 = \left((c_{i-1})^{j-1}\right)_{i,j=1}^{s}$ and I_s is the identity matrix of dimension s.

Adapted peer methods

Fitting space

$$\mathcal{F} = \left\{1, t, t^2, \ldots, t^K, e^{\pm\mu t}, t\,e^{\pm\mu t}, t^2 e^{\pm\mu t}, \ldots, t^P e^{\pm\mu t}\right\},$$

$\mu = \mathrm{i}\omega$, where $\omega \in \mathbb{R}$ is the oscillating frequency of the problem.

Linear difference operator

$$\mathcal{L}_i[h, \mathbf{w}]\, y(t) = y(t + c_i\, h) - \sum_{j=1}^{s} b_{ij}\, y(t + (c_j - 1)\, h) - h \sum_{j=1}^{s} a_{ij}\, y'(t + (c_j - 1)\, h)$$

$$- h \sum_{j=1}^{i} r_{ij}\, y'(t + c_j\, h), \quad i = 1, \ldots, s,$$

where $\mathbf{w}$ contains the coefficients of the method.

The method is adapted to the fitting space $\mathcal{F}$ if

$$\mathcal{L}_i[h, \mathbf{w}]\, y(t) = 0, \quad \forall y \in \mathcal{F}$$

$$Y_{n+1} = (B(w) \otimes I_d)\, Y_n + h\,(A(w) \otimes I_d)\, F(Y_n) + h\,(R(w) \otimes I_d)\, F(Y_{n+1})$$

Fitting space for EF peer methods - s odd

$$\mathcal{F} = \left\{ e^{\pm \mu t},\, t\, e^{\pm \mu t},\, t^2 e^{\pm \mu t}, \ldots,\, t^{(s-1)/2} e^{\pm \mu t} \right\},$$

$\mu = i\omega$, where $\omega \in \mathbb{R}$ is the oscillating frequency of the problem.

Theorem: Order conditions for EF peer methods - s odd *

The EF peer method has order $p = s$ if

$$B\, v_1 = v_2 + w^2\, A\, v_3 + w^2\, R\, v_4, \tag{6a}$$

$$A \;\; = (F_1 - B\, F_2 - R\, F_4)\; F_3^{-1}, \tag{6b}$$

with $w = \omega h$, $v_i = v_i(c, w) \in \mathbb{R}^s$, $\quad F_i = F_i(c, w) \in \mathbb{R}^{s \times s}$.

*Conte, Paternoster, Mohammadi, Moradi 2019; Conte, D'Ambrosio, Moccaldi, Paternoster 2018; Cardone, Conte, D'Ambrosio, Paternoster 2017.

Dependence of the coefficients on the frequency

$$F_1(c,w)=\begin{bmatrix} c_1\,\eta_0\left(-c_1^2w^2\right) & \dots & \frac{1}{2^P}\,c_1^{2P}\,\eta_{P-1}\left(-c_1^2\,w^2\right) & \frac{1}{2^P}\,c_1^{2P+1}\,\eta_P\left(-c_1^2\,w^2\right)\\ \vdots & & \vdots & \vdots \\ c_s\,\eta_0\left(-c_s^2w^2\right) & \dots & \frac{1}{2^P}\,c_s^{2P}\,\eta_{P-1}\left(-c_s^2\,w^2\right) & \frac{1}{2^P}\,c_s^{2P+1}\,\eta_P\left(-c_s^2\,w^2\right)\end{bmatrix},$$

with $P=\frac{s}{2}-1$, $w=\omega h$.

$\eta_k(Z)$ functions

[Ixaru, 1997; Ixaru and Vanden Berghe 2004; Conte, Esposito, Paternoster and Ixaru 2010]

$$\eta_{-1}(Z)=\begin{cases}\cos(|Z|^{1/2}) & Z\le 0\\ \cosh(Z^{1/2}) & Z>0\end{cases},\qquad \eta_0(Z)=\begin{cases}\sin(|Z|^{1/2})/|Z|^{1/2} & Z<0\\ 1 & Z=0\\ \sinh(Z^{1/2})/Z^{1/2} & Z>0\end{cases},$$

$$\eta_m(Z)=\frac{1}{Z}[\eta_{m-2}(Z)-(2m-1)\eta_{m-1}(Z)] \text{ for } m\ge 1,\ Z\neq 0,$$

$$\eta_m(0)=\frac{1}{(2m+1)!!} \text{ for } m\ge 1.$$

Numerical tests

Prothero-Robinson problem [Hairer and Wanner, 2002]

$$y'(t) = \lambda\,(\,y(t) - \sin(\omega\,t + t)\,) + (\omega + 1)\,\cos(\omega\,t + t), \quad t \in \left[0, \frac{\pi}{2}\right],$$

$$y(0) = 0,$$

exact solution $y(t) = \sin(\omega\,t)\,\cos(t) + \cos(\omega\,t)\,\sin(t)$.

- CL=classic, EF=exponentially fitted, IM=implicit, EX=explicit, Ps= s-stages peer

$\lambda = -1$		N			
	ω	80	160	320	640
EX CL P2	50	$3.79 \cdot 10^{-1}$	$1.05 \cdot 10^{-1}$	$2.65 \cdot 10^{-2}$	$6.60 \cdot 10^{-3}$
EX EF P2	50	$1.53 \cdot 10^{-2}$	$4.1 \cdot 10^{-3}$	$1.00 \cdot 10^{-3}$	$2.57 \cdot 10^{-4}$
EX CL P3	50	$1.39 \cdot 10^{-1}$	$1.10 \cdot 10^{-2}$	$9.42 \cdot 10^{-4}$	$8.98 \cdot 10^{-5}$
EX EF P3	50	$1.23 \cdot 10^{-4}$	$1.07 \cdot 10^{-5}$	$1.26 \cdot 10^{-6}$	$1.33 \cdot 10^{-7}$
EX CL P2	100	$3.59 \cdot 10^{-1}$	$3.02 \cdot 10^{-1}$	$9.33 \cdot 10^{-2}$	$2.47 \cdot 10^{-2}$
EX EF P2	100	$3.33 \cdot 10^{-2}$	$5.3 \cdot 10^{-3}$	$1.8 \cdot 10^{-3}$	$4.86 \cdot 10^{-4}$
EX CL P3	100	1.12	$6.92 \cdot 10^{-2}$	$2.40 \cdot 10^{-3}$	$1.22 \cdot 10^{-4}$
EX EF P3	100	$2.87 \cdot 10^{-5}$	$3.08 \cdot 10^{-5}$	$2.30 \cdot 10^{-6}$	$1.58 \cdot 10^{-8}$

- better accuracy of exponentially fitted methods
- smaller error as ω increases

What happens if we do not exactly know the frequency?

Prothero-Robinson problem [Hairer and Wanner, 2002]

$$y'(t) = \lambda\,(\,y(t) - \sin(\omega\,t + t)\,) + (\omega + 1)\,\cos(\omega\,t + t), \quad t \in \left[0, \frac{\pi}{2}\right],$$

$$y(0) = 0,$$

exact solution $y(t) = \sin(\omega\,t)\,\cos(t) + \cos(\omega\,t)\,\sin(t)$.

- coefficients of EF peer method computed in correspondence of $w = \omega(1 + \delta)h$
- CL=classic, EF=exponentially fitted, IM=implicit, EX=explicit, Ps= s-stages peer

$\lambda = -1$, $\omega = 50$	N			
	80	160	320	640
EX CL P3	$1.38e-1$	$1.09e-2$	$9.41e-4$	$8.98e-5$
EX EF P3 $\delta = 0.5$	$1.10e-2$	$6.18e-3$	$7.62e-4$	$1.09e-4$
EX EF P3 $\delta = 0.1$	$1.91e-3$	$1.70e-4$	$2.10e-5$	$2.31e-6$
EX EF P3 $\delta = 0$	$1.23e-4$	$1.07e-5$	$1.26e-6$	$1.33e-7$
IM CL P3	$7.50e-4$	$4.16e-5$	$2.43e-6$	$1.45e-7$
IM EF P3 $\delta = 0.5$	$1.60e-3$	$8.06e-5$	$3.93e-6$	$2.16e-7$
IM EF P3 $\delta = 0.1$	$2.97e-5$	$1.37e-6$	$7.33e-8$	$4.34e-9$
IM EF P3 $\delta = 0$	$1.62e-6$	$7.64e-8$	$3.41e-9$	$2.01e-10$

- accurate computation of the frequency is a crucial point

Stiff problem

Prothero-Robinson problem [Hairer and Wanner, 2002]

$$y'(t) = \lambda\,(\,y(t) - \sin(\omega\,t + t)\,) + (\omega + 1)\,\cos(\omega\,t + t), \quad t \in \left[0, \frac{\pi}{2}\right],$$
$$y(0) = 0,$$

exact solution $y(t) = \sin(\omega\,t)\,\cos(t) + \cos(\omega\,t)\,\sin(t)$.

- EF=exponentially fitted, IM=implicit, Ps= s-stages peer

$\lambda = -10^6, \omega = 50$	N		
	160	320	640
IM CL P2	$2.17e-6$	$2.16e-7$	$2.55e-8$
IM EF P2	$6.01e-8$	$7.74e-9$	$9.73e-10$
IM CL P3	$1.79e-7$	$2.51e-8$	$3.22e-9$
IM EF P3	$2.42e-10$	$3.48e-11$	$4.88e-12$

- both CL and EF explicit peer methods are unstable

Adapted IMEX peer methods for Boussinesq equation

Boussinesq equation

$$
\begin{aligned}
&h_t(x,t) = \gamma h_{xx}(x,t) - \nu h_x(x,t), \quad (x,t) \in (0,X) \times (0,T) \\
&h(x,0) = h_0(x), \quad x \in [0,X], \\
&h(0,t) = h(X,t) = f(t), \quad t \in [0,T],
\end{aligned}
$$

with periodic boundary condition $f(t) = \exp(i\omega t)$.

Numerical solution by means of adapted numerical methods

- Spatial semi-discretization: adapted finite differences
- Numerical solution of the system of ODEs in time: adapted IMEX peer methods

Spatial semi-discretization: adapted finite differences

$$h_t(x,t) = \gamma h_{xx}(x,t) - \nu h_x(x,t)$$

- **Profile of the solution:** $h(x,t) = \exp((\alpha + i\beta)x) \cdot \exp(i\omega t)$
- $\zeta = \alpha + i\beta \in \mathbb{C}$, $z = \zeta \Delta x$.
- **Fitting space:** $\mathcal{F} = \{1,\ exp(\zeta x), xexp(\zeta x)\}$.

Discretization of the diffusion term*

$$h_{xx}(x_n,t) \simeq \frac{a_0 h(x_{n-1},t) + a_1 h(x_n,t) + a_2 h(x_{n+1},t)}{\Delta x^2},$$

with $a_0 = -\frac{ze^z(2-2e^z+ze^z)}{(e^z-1)^2}$, $a_1 = -\frac{z(2-2e^{2z}+z+ze^z)}{(e^z-1)^2}$, $a_2 = -\frac{z(2-2e^z+z)}{(e^z-1)^2}$.

- the coefficients depend on $z = (\alpha + i\beta)\Delta x$
- $z \to 0$: the coefficients tend to classical ones $a_0 = a_2 = 1$, $a_1 = -2$
- EF finite differences preserve the accuracy of classical finite differences

*Cardone, D'Ambrosio, Paternoster, Exponentially fitted IMEX methods for advection-diffusion problems, JCAM, 2017.

Spatial semi-discretization: adapted finite differences

$$h_t(x,t) = \gamma h_{xx}(x,t) - \nu h_x(x,t)$$

- **Profile of the solution:** $h(x,t) = \exp((\alpha + i\beta)x) \cdot \exp(i\omega t)$
- $\zeta = \alpha + i\beta \in \mathbb{C}$, $z = \zeta \Delta x$.
- **Fitting space:** $\mathcal{G} = \{1,\ exp(\zeta x)\}$.

Discretization of the advection term

$$h_x(x_n,t) \simeq \frac{b_0 h(x_{n-1},t) + b_1 h(x_n,t)}{\Delta x},$$

with $b_0 = \frac{z}{(e^{-z}-1)}$, $b_1 = -\frac{z}{(e^{-z}-1)}$.

- the coefficients depend on $z = (\alpha + i\beta)\Delta x$
- $z \to 0$: the coefficients tend to classical ones $b_0 = -1$, $b_1 = 1$
- EF finite differences preserve the accuracy of classical finite differences

Spatially semi-discretized system

$$\mathbf{h}'(t) = \mathcal{A}(z)\mathbf{h}(t) + \mathcal{B}(z)\mathbf{h}(t) + g(z,t),$$

- $x_j = j\Delta x$, $j = 0, \ldots, M-1$, $x_0 = 0$, $x_{M-1} = X$,
- $z = (\alpha + i\beta)\Delta x$,
- $\mathbf{h}(t) = [h(x_i, t)]_{i=1}^{d}$, $d = M - 2$.
- $\mathcal{A}(z) = \frac{\gamma}{\Delta x^2}\mathtt{diag}(a_0, a_1, a_2)$, $\mathcal{B}(z) = -\frac{\nu}{\Delta x}\mathtt{diag}(b_0, b_1, 0)$, tridiagonal matrices of dimension d
- $g(z,t) = \left(\frac{\gamma a_0}{\Delta x^2} - \frac{\nu b_0}{\Delta x}\right) f(t)\mathbf{e_1} + \frac{\gamma a_2}{\Delta x^2} f(t)\mathbf{e_d}$

Why EF IMEX peer methods?

- oscillatory solution
- stiffness in diffusion term
- great dimension d

- $\Delta x = \frac{X}{M-1}$, $\Delta t = \frac{T}{N}$, $z = (\alpha + i\beta)\Delta x$, $w = \omega \Delta t$

Spatially semi-discretized system

ODEs system of dimension $d = M - 2$

$$\mathbf{h}'(t) = \mathcal{A}(z)\mathbf{h}(t) + \mathcal{B}(z)\mathbf{h}(t) + g(z,t),$$

- $A(w)$, $B(w)$, $R(w)$: s-stage IM EF peer method
- $\hat{A}(w)$, $\hat{B}(w)$, $\hat{R}(w)$: s-stage EX EF peer method with $\hat{B}(w) = B(w)$

EF IMEX peer methods [classical IMEX peer: Soleimani et. al, 2017]

$$Y_{n+1} = (B(w) \otimes I_d)\, Y_n + \Delta t\, (A(w) \otimes I_d)\, F(Y_n) + \Delta t\, (R(w) \otimes I_d)\, F(Y_{n+1}) + \Delta t\, (\hat{A}(w) \otimes I_d)\, G(Y_n) + \Delta t\, (\hat{R}(w) \otimes I_d)\, G(Y_{n+1})$$

- $F(Y_n) = (I_s \otimes \mathcal{A}(z))\, Y_n$ - stiff term
- $F(G_n) = (I_s \otimes \mathcal{B}(z))\, Y_n + g(z, t_n + ch)$ - nonstiff term

- peer method with s stages: accuracy $O(\Delta t^s) + O(\Delta x)$

Numerical tests

Boussinesq equation

$$h_t(x,t) = \gamma h_{xx}(x,t) - \nu h_x(x,t), \quad (x,t) \in [0,X] \times [0,T]$$
$$h(x,0) = h_0(x), \quad x \in [0,X],$$
$$h(0,t) = h(X,t) = f(t), \quad t \in [0,T],$$

with periodic boundary condition $f(t) = \exp(i\omega t)$.

Parameters: $\gamma = 5$, $\nu = 2$, $\omega = 2$, $T = X = 10$, $\Delta x = \Delta t = 0.1$

- CL=classic, EF=exponentially fitted, IM=implicit
- FD=finite differences, Ps= s-stages peer

Space/Time	IM CL P2	IMEX CL P2	IM EF P2	IMEX EF P2
CL FD	$6.05e-03$	$2.59e-02$	$5.04e-03$	$5.04e-03$
EF FD	$9.84e-03$	$2.71e-02$	$6.67e-14$	$6.57e-14$

- both CL and EF explicit time solvers are unstable

Numerical tests

Boussinesq equation

$$h_t(x,t) = \gamma h_{xx}(x,t) - \nu h_x(x,t), \quad (x,t) \in [0,X] \times [0,T]$$
$$h(x,0) = h_0(x), \quad x \in [0,X],$$
$$h(0,t) = h(X,t) = f(t), \quad t \in [0,T],$$

with periodic boundary condition $f(t) = \exp(i\omega t)$.

Parameters: $\gamma = 5$, $\nu = 2$, $\omega = 20$, $T = X = 10$, $\Delta x = \Delta t = 0.1$

- CL=classic, EF=exponentially fitted, IM=implicit
- FD=finite differences, Ps= s-stages peer

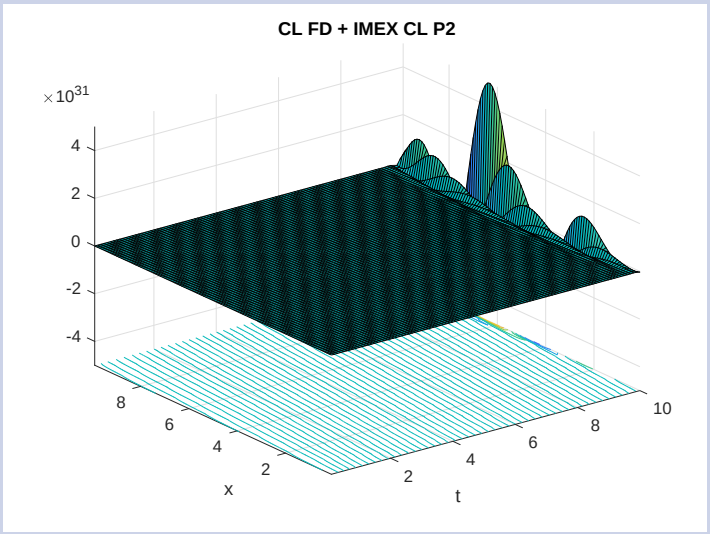
Space/Time	IM CL P2	IMEX CL P2	IM EF P2	IMEX EF P2
CL FD	$1.03e-01$	$1.53e+00$	$3.85e-03$	$3.85e-03$
EF FD	$3.98e-01$	$1.53e+00$	$5.36e-15$	$4.00e-15$

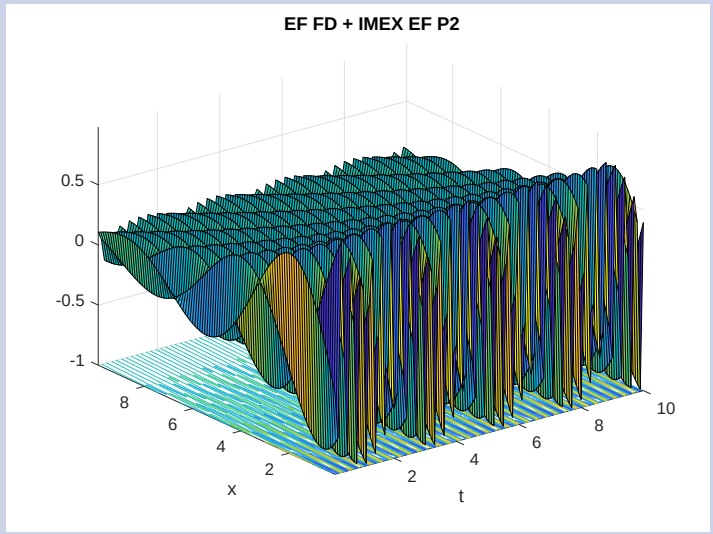
- both CL and EF explicit time solvers are unstable

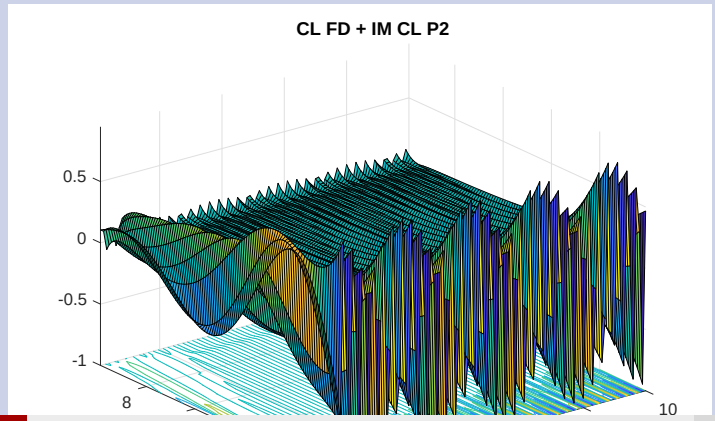
Numerical tests

Parameters: $\gamma = 5$, $\nu = 20$, $\omega = 20$, $T = X = 10$, $\Delta x = \Delta t = 0.1$

- CL=classic, EF=exponentially fitted, IM=implicit
- FD=finite differences, Ps= s-stages peer







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Stochastic Volterra integral equations

Conclusions

- ϑ-method for SVIEs
  - same order of convergence of Euler-Maruyama method
  - better stability properties
- improved ϑ-method for SVIEs
  - same order convergence of ϑ-method
  - further improvement in stability properties with respect to convolution test equation

Future work

- improve stability regions of higher order methods for SVIEs (e.g. Milstein type)
- analysis of nonlinear stability properties

Ordinary differential equations with oscillatory solution

Conclusions

- ODEs with oscillatory solution
  - explicit EF peer methods for nonstiff problems
  - implicit EF peer methods for stiff problems
  - order $p = s$
  - parallel methods for diagonal R
- linear advection-diffusion problems with oscillatory solution
  - EF finite differences
  - EF IMEX peer methods
  - accuracy $O(\Delta t^s)+O(\Delta x)$

Future work

- stability analysis
- application to nonlinear problems
- parallel implementation

Thank you for your attention